1. Let $\phi : l^2 \to \mathbb{C}$ be a linear functional.
   Consider the standard basis $\{e_k\}$ of l^2.

   To show that, there exists $y \in l^2$ such that $\phi(x) = \langle x, y\rangle$ for all $x \in l^2$.
   Let $x \in l^2$.
   For $N \in \mathbb{N}$, we define $S_N = \sum_{j=1}^{N} \langle x, e_j\rangle e_j$.
   $\therefore \quad \|x - S_N\| \to 0$ as $N \to \infty$.
   Since, ϕ is linear we get $\phi(S_N) = \sum_{j=1}^{N} \langle x, e_j\rangle \phi(e_j)$.
   ϕ is bounded
   $\Rightarrow \phi(x) = \lim\limits_{N\to\infty} \phi(S_N) = \sum_{j=1}^{\infty} \langle x, e_j\rangle \phi(e_j)$.
   Now, Choose $z = \sum_{j=1}^{N} \overline{\phi(e_j)} e_j$, then $\phi(z) = \sum_{j=1}^{N} |\phi(e_j)|^2$.
   $\sum_{j=1}^{N} |\phi(e_j)|^2 = \phi(z) \leq \|\phi\|\|z\| = \|\phi\| \left(\sum_{j=1}^{N} |\phi(e_j)|^2\right)^{\frac{1}{2}}$
   $\Rightarrow \left(\sum_{j=1}^{N} |\phi(e_j)|^2\right)^{\frac{1}{2}} \leq \|\phi\|$.
   As, N is arbitrary, therefore $\{\phi(e_j)\}$ is square summable and we can define $y = \sum_{j=1}^{\infty} \overline{\phi(e_j)} e_j$.
   It is easy to check that $\phi(x) = \langle x, y\rangle$.

2. Let $T \in \mathcal{B}(X)$.

   The approximate point spectrum of T is denoted by $\sigma_{AP}(T)$ and is defined as

   $\sigma_{AP}(T) := \{\lambda \in \mathbb{C} : (\lambda I - T)$ is not bounded below$\}$.

   Clearly $\sigma_{AP}(T) \subseteq \sigma(T)$, spectrum of T.

   As the spectrum set $\sigma(T)$ is bounded by $\|T\|$, it remains to show that $\sigma_{AP}(T)$ is closed i.e to show that it's complement is open in $\mathbb{C}$.

   Let $\lambda \in \mathbb{C}\backslash\sigma_{AP}(T)$.

   Therefore, there exists an $\alpha > 0$ such that $\|(T - \lambda I)x\| \geq \alpha\|x\|$ for all $x \in X$.

   Let $\mu \in \mathbb{C}$ be arbitrary.

   $\therefore \quad \|(T - \lambda I)x\| \leq \|(T - \mu I)x\| + \|(\mu - \lambda)x\|$

   $\Rightarrow \|(T - \mu I)x\| \geq (\alpha - |\mu - \lambda)|\|x\|$.

   Therefore, $(T - \mu I)$ is bounded below for all $\mu \in \mathbb{C}$ such that $\alpha > |\mu - \lambda)|$.

   This shows that, the open ball $B_\alpha(\lambda) \subseteq \mathbb{C}\backslash\sigma_{AP}(T)$.

   Therefore $\mathbb{C}\backslash\sigma_{AP}(T)$ is open and we are done.

3. We know that, if E is a convex subset of a locally convex space X, then the weak closure of E is same as the original closure of E (see the book "Functional Analysis" by W. Rudin Theorem 3.12).
   Consider $E = \text{span}\{x_n\}$, which is convex set in the Banach space X.
   Let the sequence $\{x_n\} \subseteq E$ converges to x weakly i.e x is in the weak closure of E. Therefore, from the above theorem x is in the weak closure of E i.e there is a sequence $\{y_n\}$ in E such that $y_n \to x$ in the norm topology.

4. Let X be a Banach space.

   Let $J_X : X \to X^{**}$ be the cannonical map i.e $J_X(x)(f) = f(x)$ for $x \in X$ and $f \in X^*$.

   Let us assume that X if reflexive i.e the map J is surjective.

Consider the cannonical map $J_{X^*} : X^* \to X^{***}$.
We have to show that for $F \in X^{***}$ there exists an $f \in X^*$ such that $F = J_{X^*}(f)$.
For any $\phi \in X^{**}$ there is $x_\phi \in X$ such that $J_X(x_\phi) = \phi$ as J is onto.
Now, $J_{X^*}(F \circ J_X)(\phi) = \phi(F \circ J_X) = J_X(x_\phi)(F \circ J_X) = (F \circ J_X)(x_\phi) = F(\phi)$.
Therefore, we get $F = J_{X^*}(F \circ J_X)$ and we are done.
Now, conversely let J_{X^*} is surjective.
Let us assume that X is not reflexive i.e there is a $\phi(\neq 0) \in X^{**} \backslash J_X(X)$.
As J_X is an isometry so $J_X(X)$ is closed subspace in X^{**} and therefore we can get $F \in X^{***}$ such that $F\big|_{J_X(X)} = 0$ and $F(\phi) \neq 0$ (see Lemma 4.6-7 in Kreyszig).
But as J_{X^*} is surjective, there is an $f \in X^*$ such that $F = J_{X^*}(f)$.
Now, for $x \in X$ we have $f(x) = J_X(x)(f) = J_{X^*}(f)(J_X(x)) = F(J_X(x)) = 0$, which is a contradiction.

5. Let the map $T : X \to Y$ is weak continuous where X, Y are banch spaces.
   By closed graph theorem if we can show that graph of T is closed then it is continuos.
   Let the sequence $(x_n, T(x_n))$ is convergent and converges to (x, y) where $\{x_n\}$ is a sequence in X, $x \in X, y \in Y$.
   Therefore, we have to show that $y = T(x)$.
   $x_n \to x$ in norm $\Rightarrow x_n \to x$ weakly and as T is weakly continuous it follows that $T(x_n) \to T(x)$ weakly.
   But from our assumption $T(x_n) \to y$ weakly, therefore from uniqueness of weak limit in a Banach space we get $y = T(x)$ and therefore T is continuous.

6. For the solution of this question, see Theorem 2 (page number 170) and Theorem 18 (page number 18) from the book "Notes on Functional Analysis" by Rajendra Bhatia.

7. Let $A, B \in \mathcal{B}(\mathcal{H})$ and A is positive such that $AB = BA$.
   Now, $A^2B = A(AB) = A(BA) = (AB)A = (BA)A = BA^2$ and in this way one can show that $A^nB = BA^n$ for any $n \in \mathbb{N}$.
   Therefore, $p(A)B = Bp(A)$ for any polynomial p.
   Let $\sqrt{A}$ be the square root of A.
   Let $p_n(A)$ be a sequence of polynomials in A which converges to $\sqrt{A}$ strongly.
   (For the construction of the sequence of polynomials, see Theorem 3, page number-157 from the book "Notes on Functional Analysis" by Rajendra Bhatia).
   As $p_n(A)B = Bp_n(A)$ and $p_n(A) \to \sqrt{A}$ strongly we have $\sqrt{A}B = B\sqrt{A}$.

8. This is "Hahn-Banach Theorem" and one can find it from any Functional Analysis book (ex- "Notes on Functional Analysis" by Rajendra Bhatia).

9. Let P_1, P_2 be two orthogonal projections on a Hilbert space $\mathcal{H}$.
   $\therefore$ We have $P_i^2 = P_i^* = P_i$ for $i = 1, 2$.
   Also we have $(P_1 + P_2)^* = (P_1 + P_2)$.

$$\begin{aligned}\text{Therefore } (P_1 + P_2) \text{ is an orthogonal projection } &\iff (P_1 + P_2)^2 = (P_1 + P_2) \\ &\iff P_1P_2 + P_2P_1 = 0\end{aligned}$$

   .
   So,if $P_1P_2 = 0$, then $P_1 + P_2$ is an orthogonal projection.
   Now, conversely let $P_1 + P_2$ is a projection then $P_1P_2 + P_2P_1 = 0$.

Multiplying the above thing by $(I - P_1)$ we get,

$$
\begin{aligned}
&(I - P_1)(P_1 P_2 + P_2 P_1) = 0 \\
&\Rightarrow (I - P_1) P_2 P_1 = 0 \\
&\Rightarrow P_2 P_1 = P_1 P_2 P_1 \\
&\Rightarrow P_1 P_2 = (P_2 P_1)^* = (P_1 P_2 P_1)^* = P_1 P_2 P_1 = P_2 P_1 \\
&\Rightarrow P_1 P_2 = 0 \ \text{ as } \ P_1 P_2 + P_2 P_1 = 0
\end{aligned}
$$